Some interpolation properties and tensor product stability of Stolz Mappings

Nicolae Tiţa

1 Introduction

Let X be a Banach space and let $T \in \mathbf{L}(X)$ be a linear and bounded operator $T : X \to X$.
By $\{a_n(T)\}$ and $\{e_n(T)\}$ we denote the sequences of the approximation numbers and entropy numbers of T (dyadic entropy numbers) [2], [5], [8].
The class of Stolz mappings has been defined by K. Iseki, see [4], as follows:

$$L_{STOL,p}(X) = \left\{ T : \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \cdots + \alpha_n} \sum_{i=1}^{n} \alpha_i a_i(T) \right)^p \right)^{\frac{1}{p}} < \infty \right\}, \; 0 < p < \infty$$

where $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$.
In [4] is proved that if $\lim_{n\to\infty} \alpha_n \neq 0$ then

$$L_{STOL,p}(X) = L_p(X) = \left\{ T : \left(\sum_{n=1}^{\infty} a_n^p(T) \right)^{\frac{1}{p}} < \infty \right\}$$

Remark 1.1. The application $\Phi_n^{\alpha} : \{a_i(T)\}_1^n \to \sum_{i=1}^{n} \alpha_i a_i(T)$ is a symmetric norming function, which is not equivalent with the maximal function $\Phi_1 : \{a_i(T)\}_1^n \to \sum_{i=1}^{n} a_i(T)$ if $\lim_{n\to\infty} a_n(T) = 0$ [2], [8]. In this way we remark that the class of Stolz mappings is a particular case of the classes

$$L_{M_\Phi,p}(X) = \left\{ T : \left(\sum_{n=1}^{\infty} \left(\frac{\Phi\left(\{a_i(T)\}_1^n\right)}{\Phi(n)} \right)^p \right)^{\frac{1}{p}} < \infty \right\}, \; 0 < p < \infty$$

where $\Phi(n) = \Phi(\underbrace{1, \ldots, 1}_{n}, 0, 0, \ldots)$ and Φ is a symmetric norming function. The classes $L_{M_\Phi,p}$ has been presented in [6] and in the other lectures.

For the properties of the function Φ and $\{a_n(T)\}$, $\{e_n(T)\}$ it can see [2], [8].
It is known that $L_{STOL,p}$ is an operator ideal (quasinormed). Also $L_{M_\Phi,p}$ is a quasinormed operator ideal, because

$$\sum_{n=1}^{k} a_n(S+T) \leq 2 \sum_{n=1}^{k} \left[a_n(S) + a_n(T) \right], \; k = 1, 2, \ldots$$

[8], [9]. In the following we present some properties for $L_{STOL,p}$.

2 Results

Theorem 2.1. *If $S \in L_{STOL,s,q}(X)$ and $T \in L_{STOL,t,r}(X)$, then $ST \in L_{STOL,p}(X)$, where $1 = \frac{1}{s} + \frac{1}{t}$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, $1 \le p < \infty$ and*

$$L_{STOL,s,q}(X) = \left\{ T : \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \cdots + \alpha_n} \sum_{i=1}^{n} \alpha_i a_i^s(T) \right)^{\frac{q}{s}} \right)^{\frac{1}{q}} < \infty \right\}$$

Proof.

$$\begin{aligned} \|ST\|_{STOL,p} &= \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \cdots + \alpha_n} \sum_{i=1}^{n} \alpha_i a_i(ST) \right)^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^{\infty} \left(\frac{\Phi^{\alpha}\left(\{a_i(ST)\}_1^n\right)}{\Phi^{\alpha}(n)} \right)^p \right)^{\frac{1}{p}} \end{aligned}$$

where $\Phi^{\alpha}(n) = \Phi^{\alpha}(\underbrace{1, \ldots, 1}_{n}, 0, 0, \ldots)$. Since

$$\sum_{i=1}^{k} a_i(ST) \le 2 \sum_{i=1}^{k} \left[a_i(S) \cdot a_i(T) \right], \ k = 1, 2, \ldots$$

and Φ^{α} is a symmetric norming function it follows:

$$\begin{aligned} \|ST\|_{STOL,p} &\le 2 \left(\sum_{n=1}^{\infty} \left(\frac{\Phi^{\alpha}\left(\{a_i(S)a_i(T)\}_{i=1}^n\right)}{\Phi^{\alpha}(n)} \right)^p \right)^{\frac{1}{p}} \\ &\le 2 \left(\sum_{n=1}^{\infty} \left(\frac{\Phi^{\alpha}_{(s)}\left(\{a_i(S)\}_{i=1}^n\right) \cdot \Phi^{\alpha}_{(t)}\left(\{a_i(T)\}_{i=1}^n\right)}{\Phi^{\alpha}_{(s)}(n) \cdot \Phi^{\alpha}_{(t)}(n)} \right)^p \right)^{\frac{1}{p}} \\ &\le 2 \left(\sum_{n=1}^{\infty} \left(\frac{\Phi^{\alpha}_{(s)}\left(\{a_i(S)\}_{i=1}^n\right)}{\Phi^{\alpha}_{(s)}(n)} \right)^q \right)^{\frac{1}{q}} \cdot \left(\sum_{n=1}^{\infty} \left(\frac{\Phi^{\alpha}_{(t)}\left(\{a_i(T)\}_{i=1}^n\right)}{\Phi^{\alpha}_{(t)}(n)} \right)^r \right)^{\frac{1}{r}} \end{aligned}$$

where $1 = \frac{1}{s} + \frac{1}{t}$ [1], $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Hence $ST \in L_{STOL,p}(X)$. ■

Theorem 2.2. *The classes $L_{STOL,p}(X)$ are tensor product stable for all tensor norms, if the sequence $(\alpha_n)_n$ is such that $\alpha_{n^2} \le \frac{C}{n}\alpha_n$, $(\forall)n = 1, 2, \ldots$ and C is a constant (depending only of the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$).*

Proof. The proof is a corollary of the inequality:

$$\sum_{n=1}^{k} \alpha_n a_n(S \otimes T) \le C(\alpha) \sum_{n=1}^{k} \alpha_n \left[a_n(S) + a_n(T) \right]$$

[1] We remark that $\Phi^{\alpha}_{(s)}(a_i(S)) = \left(\sum_{i=1}^{n} \alpha_i a_i^s(S) \right)^{\frac{1}{s}}$

[7], which is true for all (fixed!) $S, T \in \mathbf{L}(X)$.
We obtain:

$$\begin{aligned}\|S \otimes T\|_{STOL,p} &\leq C(\alpha) \cdot \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \cdots + \alpha_n} \sum_{i=1}^{n} \alpha_i \left[a_i(S) + a_i(T) \right] \right)^p \right)^{\frac{1}{p}} \\ &\leq C(\alpha, p) \left[\|S\|_{STOL,p} + \|T\|_{STOL,p} \right] < \infty\end{aligned}$$

■ Now we present a interpolation theorem of Riesz-Thorin type for the ideals

$$L_{STOL,p}^{(e)}(X) = \left\{ T : \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \cdots + \alpha_n} \sum_{i=1}^{n} \alpha_i e_i(T) \right)^p \right)^{\frac{1}{p}} < \infty \right\}$$

where $\{e_i(T)\}$ is the sequence of the dyadic entropy numbers.
If (X_0, X_1) is an interpolation couple of two normed spaces and $(X_0, X_1)_{\theta,q}$ is the interpolation space, $\theta \in (0,1)$, $0 < q < \infty$, it is known that:

$$e_{2n-1}\left(T : (X_0, X_1)_{\theta,q} \to X \right) \leq 2 e_n \left(T : X_0 \to X \right)^{1-\theta} \cdot e_n \left(T : X_1 \to X \right)^{\theta}$$

for all normed spaces X.
Since

$$\|T\|_{STOL,p}^{(e)} \sim \|T\|^{*(e)}_{STOL,p} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \cdots + \alpha_n} \sum_{i=1}^{n} \alpha_i e_{2i-1}(T) \right)^p \right)^{\frac{1}{p}}$$

we obtain

$$\left\| T : (X_0, X_1)_{\theta,q} \to X \right\|_{STOL,p}^{(e)} \leq$$

$$\begin{aligned}&\leq C \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \cdots + \alpha_n} \sum_{i=1}^{n} \alpha_i e_i \left(T : X_0 \to X \right)^{1-\theta} \cdot e_i \left(T : X_1 \to X \right)^{\theta} \right)^p \right)^{\frac{1}{p}} \\ &\leq C \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \cdots + \alpha_n} \left(\sum_{i=1}^{n} \alpha_i e_i \left(T : X_0 \to X \right)^{p_1} \right)^{\frac{1-\theta}{p_1}} \left(\sum_{i=1}^{n} \alpha_i e_i \left(T : X_1 \to X \right)^{p_2} \right)^{\frac{\theta}{p_2}} \right) \\ &\leq C \left[\sum_{n=1}^{\infty} \frac{1}{\alpha_1 + \cdots + \alpha_n} \left(\sum_{i=1}^{n} \alpha_i e_i \left(T : X_0 \to X \right)^{p_1} \right)^{\frac{r}{p_1}} \right]^{\frac{1-\theta}{r}} \cdot \\ &\quad \cdot \left[\sum_{n=1}^{\infty} \frac{1}{\alpha_1 + \cdots + \alpha_n} \left(\sum_{i=1}^{n} \alpha_i e_i \left(T : X_1 \to X \right)^{p_2} \right)^{\frac{s}{p_2}} \right]^{\frac{\theta}{s}}\end{aligned}$$

where $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\theta \in (0,1)$, $0 < p_1 < p_2 < \infty$ and $1 = \frac{1-\theta}{r} + \frac{\theta}{s}$.
Hence we obtain:

Proposition 2.3. $L_{STOL,p_1,r}^{(e)}(X_0, X) \cap L_{STOL,p_2,s}^{(e)}(X_1, X) \subseteq L_{STOL,p}^{(e)}\left((X_0, X_1)_{\theta,q}, X \right)$ *if* $0 < p_1 < p_2 < \infty$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $1 = \frac{1-\theta}{r} + \frac{\theta}{s}$, $\theta \in (0,1)$, $0 < r < s < \infty$.

Remark 2.4. All results are valid if the function Φ^{α} is replaced by an other function Φ.

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